Generalized Bol-Moufang Identities of Loops and Quasigroups

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1. Introduction

A *quasigroup* is a set G together with a binary operation $*$ that has the Latin Square Property; that is, for all $a \in G$, the maps $L(a) : G \to G$ and $R(a) : G \to G$ defined by $[L(a)](x) = a * x$ and $[R(a)](x) = x * a$, respectively, are bijective.

A quasigroup $(G, *)$ has the left and right cancellation properties. This means that for $x, y, z \in G$, if $x * z = y * z$, then $x = y$, and if $z * x = z * y$, then $x = y$. These properties result from the bijectivity of $L(a)$ and $R(a)$.

A *neutral element*, or an *identity*, is an element e in G such that $e * x = x = x * e$ for all $x \in G$. A *left loop* is a quasigroup $(G, *)$ for which $*$ has a left neutral element. That is, there exists $e_l \in G$ such that $e_l * x = x$ for all $x \in G$. A quasigroup is a *right loop* if it has a right neutral element; that is there exists $e_r \in G$ such that $x * e_r = x$ for all $x \in G$. It is a *loop* if it has both a left and a right neutral element. If this is the case, then the left and right neutral elements are in fact the same element.

Neither loops nor quasigroups are guaranteed to have the associativity property that groups have; that is, for $x, y, z \in G$ we do not necessarily know that $x(yz) = (xy)z$. This is one property for which we tested each of the identities. We also tested for the left alternative, right alternative, and flexible properties as defined below for all $x, y \in G$.

- Left alternative: $(xx)y = x(xy)$
- Right alternative: $(xy)y = x(yy)$
- Flexible: $x(yx) = (xy)x$

In a quasigroup $(G, *)$, $*$ is often referred to as the *principal operation*. However, we are able to define other binary operations on G as well, most notably $\backslash : G \to G$ defined by $x \backslash z = y$ if and only if $x * y = z$, and $/ : G \to G$ defined by $z/y = x$ if and only if $x * y = z$, for all $x, y, z \in G$. As a result of these operations, the *variety of quasigroups* consists of the universal algebras $(G, *, \backslash, /)$ whose binary operations $*, \backslash, /$ satisfy the following properties for all $x, y \in G$:

$$(x * y)/y = x,\ x\backslash(x * y) = y,\ (x/y) * y = x,\ y * (y\backslash x) = x.$$

A variety of quasigroups is said to be of *Bol-Moufang type* if two of its three variables occur once on each side of an identity, the third variable occurs twice on each side, and the ordering of the variables on each side is the same. In this paper, we considered generalizations of this notion. First, we classified identities with four distinct variables on each side. Then, we looked at identities with five variables, four distinct and one repeated on each side.

The notion of the variety of quasigroups allowed us to work extensively with automated theorem proving. In particular, we used the automated theorem prover Prover9 to find which implications held among our identities. We were able to "humanize" some of our results for clarity of understanding, and several of these proofs are expounded in this paper. We were also able to find counterexamples by the use of the finite model builder Mace4.

2. Varieties with Four Distinct Variables

Let x, y, z, u be elements of a set G used to construct identities of Bol-Moufang type. Because the identities are of Bol-Moufang type, the elements will appear in the same order on both sides. Without loss of generality, we assume the elements appear in the order shown below. For convenience, we write xy instead of $x * y$.

The ordering A shown below is the only way to order these four distinct variables. There are five ways to parenthesize these elements, which are denoted by $1-5$ below. We adapted this notation from Phillips and Vojtěchovský in [1] and [2].

$$\begin{array}{c|c} \mathrm{A} & xyzu \end{array} \quad \begin{array}{c|l} 1 & 0(0(00)) \\ 2 & 0((00)0) \\ 3 & (00)(00) \\ 4 & (0(00))0 \\ 5 & ((00)0)0 \end{array}$$

For example, A1-2 is the identity $x(y(zu)) = x((yz)u)$.

We started out by analyzing the identities within a loop.

Proposition: In a loop, all of these identities reduce to associativity.

Proof. By substituting the neutral element for one of the variables in each of the identities, we see that all will reduce to the form $x(yz) = (xy)z$. □

Next, we consider the varieties of quasigroups defined by the various identities $Ai-j$. We found with easy substitution that five of these imply associativity.

- $A1-2 \Rightarrow$ *Associativity* :

  Proof. $x(y(zu)) = x((yz)u)$, so left cancelling x yields $y(zu) = (yz)u$. □

- $A1-3 \Rightarrow$ *Associativity* :

  Proof. $x(y(zu)) = (xy)(zu)$. Let $zu = a$. Then $x(ya) = (xy)a$. □

- $A2-4 \Rightarrow$ *Associativity* :

  Proof. $x((yz)u) = (x(yz))u$. Let $yz = a$. Then $x(au) = (xa)u$. □

- $A3-5 \Rightarrow$ *Associativity* :

  Proof. $(xy)(zu) = ((xy)z)u$. Let $xy = a$. Then $a(zu) = (az)u$. □

- $A4-5 \Rightarrow$ *Associativity* :

  Proof. $(x(yz))u = ((xy)z)u$, so right cancelling u yields $x(yz) = (xy)z$. □

Two of these identities do not imply associativity.

- $A-14 \not\Rightarrow$ *Associativity* : Counterexample:

$*$	0	1	2
0	1	0	2
1	2	1	0
2	0	2	1

  Notice that $0*(1*2)=0*0=1$, but $(0*1)*2=0*2=2$.

- $A2-5 \not\Rightarrow$ *Associativity* : Counterexample:

$*$	0	1	2
0	1	2	0
1	0	1	2
2	2	0	1

Notice that $0*(1*2)=0*2=0$, but $(0*1)*2=2*2=1$.

We claim that the remaining three identities, A1-5, A2-3, and A3-4 are equivalent to associativity, based on our output from Prover9.

- $A2-3 \Rightarrow$ *Associativity* :

  Proof. Assume that the following equations hold true:

$$x[(yz)u]=(xy)(zu) \tag{1}$$

$$x(x\backslash y)=y \tag{2}$$

$$x\backslash(xy)=y \tag{3}$$

$$(x/y)y=x \tag{4}$$

  Notice that $(2),(3)$, and (4) are due to the fact that $A2-3$ is an equational quasigroup.

  We want to show that there do not exist c_1,c_2,c_3 such that

$$(c_1c_2)c_3 \neq c_(c_2c_3) \tag{5}$$

  Making the substitutions $(x,y)\mapsto(x/y,y)$ into (3) gives us

$$(x/y)\backslash[(x/y)y]=y$$

  And using (4) we get

$$(x/y)\backslash x=y \tag{6}$$

  Making the substitutions $(x,y,z,u)\mapsto(x,y,u,u\backslash z)$ into (1) gives us

$$x[(yu)(u\backslash z)]=(xy)[u(u\backslash z)]$$

  Using (2) we get

$$x[(yu)(u\backslash z)]=(xy)z \tag{7}$$

  Making the substitutions $(x,y,z,u)\mapsto(c_1,c_2,c_3,c_3/x)$ into (7) gives us

$$c_1[(c_2(c_3/x))((c_3/x)\backslash c_3)]=(c_1c_2)c_3$$

Using (5) we get
$$c_1[(c_2(c_3/x))((c_3/x)\backslash c_3)] \neq c_1(c_2c_3)$$

Using (6) we get
$$c_1[(c_2(c_3/x))x] \neq c_1(c_2c_3) \tag{8}$$

Making the substitutions $(x, y, z, u) \mapsto (c_2, c_3/x, x, y)$ into (7) gives us
$$c_2[((c_3/x)y)(y\backslash x)] = [c_2(c_3/x)]x \tag{9}$$

Using (9) within (8) we get
$$c_1[c_2(((c_3/x)y)(y\backslash x))] \neq c_1(c_2c_3) \tag{10}$$

Making the substitutions $(c_1, c_2, c_3, x, y) \mapsto (c_1, c_2, c_3, x, x)$ into (10)
$$c_1[c_2(((c_3/x)x)(x\backslash x))] \neq c_1(c_2c_3)$$

Using (5) we get
$$c_1[c_2(c_3(x\backslash x))] \neq c_1(c_2c_3)$$

Making the substitutions $(c_1, c_2, c_3, x) \mapsto (c_1, c_2, c_3, c_3)$ gives us
$$c_1[c_2(c_3(c_3\backslash c_3))] \neq c_1(c_2c_3)$$

Using (2) we get
$$c_1(c_2c_3) \neq c_1(c_2c_3)$$

This is the contradiction that we wanted. Therefore $A - 23 \Rightarrow$ *Associativity*.

□

- $A3 - 4 \Rightarrow A1 - 5$:

Proof. We wish to show that in a quasigroup, G, if
$$(xy)(zu) = (x(yz))u \tag{11}$$

for all $x, y, z, u \in G$, then $x(y(zu)) = ((xy)z)u$. Suppose by way of contradiction that there exist $c_1, c_2, c_3, c_4 \in G$ such that
$$((c_1c_2)c_3)c_4 \neq c_1(c_2(c_3c_4)). \tag{12}$$

Since $(xy)(zu) = (x(yz))u$, we can set $z = y\backslash z$ and get $(xy)((y\backslash z)u) = (x(y(y\backslash z)))u$. By the equational quasigroup axiom $x(x\backslash y) = y$, we have that
$$(xy)((y\backslash z)u) = (xz)u. \tag{13}$$

Left-dividing by xy, we get
$$(xy)\backslash((xz)u) = (y\backslash z)u. \tag{14}$$

Now let $(x, y, z, u) \mapsto (u, x, y, (y\backslash x)z)$ in (14). Then we have
$$(ux)\backslash((uy)((y\backslash x)z)) = (x\backslash y)((y\backslash z)z). \tag{15}$$

Substitute into (13) $(x, y, z, u) \mapsto (u, y, x, z)$ and get $(uy)((y\backslash x)z) = (ux)z$. Left-dividing by (ux), we get $(ux)\backslash(uy)((y\backslash x)z) = z$.

Thus we have that the left hand side of (15) is equal to z, so the right hand side of (15) must also equal z.

i.e.

$$(x\backslash y)((y\backslash z)z) = z. \tag{16}$$

Left-dividing by $x\backslash y$, we get

$$(x\backslash y)\backslash z = (y\backslash z)z \tag{17}$$

By substituting $(x, y, z, u) \mapsto (c_1 x, x\backslash c_2, c_3, c_4)$ into (11), we get that

$$((c_1 x)(x\backslash c_2))(c_3 c_4) = ((c_1 x)((x\backslash c_2)c_3))c_4.$$

Substituting $(x, y, z, u) \mapsto (c_1, x, c_2, c_3)$ into (13), we get that

$$((c_1 x)((x\backslash c_2)c_3))c_4 = ((c_1 c_2)c_3)c_4 \tag{18}$$

Since we assumed that $((c_1 c_2)c_3)c_4 \neq c_1(c_2(c_3 c_4))$, we may say that

$$((c_1 x)(x\backslash c_2))(c_3 c_4) \neq c_1(c_2(c_3 c_4)).$$

Let $x = c_1\backslash c_1$. Then we have that

$$((c_1(c_1\backslash c_1))((c_1\backslash c_1)\backslash c_2)(c_3 c_4) \neq c_1(c_2(c_3 c_4)) \tag{19}$$

Substituting $(x, y, z) \mapsto (c_1, c_1, c_2)$ in (17) we get $((c_1(c_1\backslash c_1))((c_1\backslash c_1)\backslash c_2)(c_3 c_4) = (c_1((c_1\backslash c_1)c_2))(c_3 c_4)$.

Using this and (19) we get

$$(c_1((c_1\backslash c_1)c_2))(c_3 c_4) \neq c_1(c_2(c_3 c_4)) \tag{20}$$

Substituting $(x, y, z, u) \mapsto (c_1, c_1\backslash c_1, c_2, c_3 c_4)$ into (11) we get

$$\begin{aligned} (c_1((c_1\backslash c_1)c_2))(c_3 c_4) &= (c_1(c_1\backslash c_1))(c_2(c_3 c_4)) \\ &= c_1(c_2(c_3 c_4)) \end{aligned}$$

This contradicts (20), so we must have that $x(y(zu)) = ((xy)z)u$ for all $x, y, z, u \in G$.

□

- $A1 - 5 \Rightarrow$ *Associativity* :

Proof. We begin with A1-5, which is the identity:

$$x(y(zu)) = ((xy)z)u \tag{21}$$

If a, b, c, d are any expressions involving variables (and operations), the phrase $A1 - 5[a, b, c, d]$ means "substitute a for x, b for y, c for z and d for u in A-15".

This is the key result.

Lemma

A1-5 implies the identity

$$x[((yz)u)(wv)] = x[(y(zu))(wv)]$$

Note that the Lemma immediately implies associativity, since we can cancel x off the left, and then cancel wv off the right, leaving us with $(yz)u = y(zu)$.

Proof of Lemma.

We begin with the expression:

$$x[y(z(u(wv)))] \tag{22}$$

Apply $A1-5[y, z, u, wv]$ to obtain the left hand side of the equation in the Lemma.

Next, apply $A1 - 5[x, y, z, u(wv)]$ to (22) to obtain

$$((xy)z)[u(wv)]$$

Apply $A1 - 5[(xy)z, u, w, v]$ to get

$$[[((xy)z)u]w]v$$

Apply $A1 - 5[x, y, z, u]$ to get

$$[[x(y(zu))]w]v$$

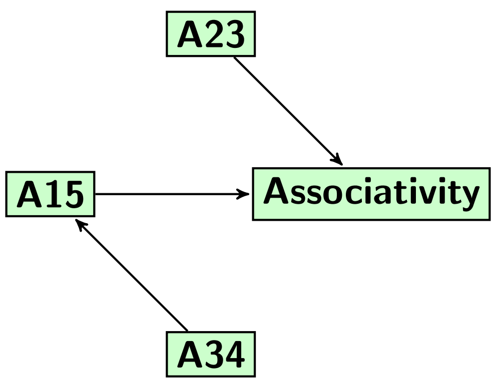
and finally apply $A1 - 5[x, y(zu), w, v]$ to get

$$x[(y(zu))(wv)]$$

which is the right hand side of the equation in the Lemma.

□

Now we have shown the implications in the diagram below. Note that since $A2 - 3$ implies associativity, it also implies $A1 - 5$ and $A3 - 4$. Thus, we have that these three identities are equivalent to associativity.



3. Varieties with Five Variables

The orderings $A - J$ shown below are the ten distinct ways to order five variables with four distinct and one repeated. There are 14 possible parenthesization patterns to consider for each of the variable orderings.

A	$xxyzu$
B	$xyxzu$
C	$xyzxu$
D	$xyzux$
E	$xyyzu$
F	$xyzyu$
G	$xyzuy$
H	$xyzzu$
I	$xyzuz$
J	$xyzuu$

1	0(0(0(00)))
2	0(0((00)0))
3	0((00)(00))
4	0((0(00))0)
5	0(((00)0)0)
6	(00)(0(00))
7	(00)((00)0)
8	((00)0)(00)
9	(0(00))(00)
10	(0(0(00)))0
11	(0((00)0))0
12	((00)(00))0
13	((0(00))0)0
14	(((00)0)0)0

We tested each of these identities within a loop. Because a loop has a neutral element, we were able to substitute the neutral element in for one of the variables to arrive at associativity for many of the identities. For those identities that were not plausible to work by hand, we were aided by Prover9. We tested each identity for associativity, the left alternative property, the right alternative property, and the flexible property. The following summarizes our results. Each box contains identities that imply the properties that are stated at the top of that box.

Associative:

- *A*: A1-2, A1-3, A1-4, A1-5, A1-7, A1-10, A1-11, A1-12, A1-13, A1-14, A2-3, A2-4, A2-5, A2-6, A2-8, A2-9, A2-11, A3-4, A3-5, A3-6, A3-7, A3-8, A3-9, A3-10, A3-11, A3-12, A3-13, A3-14, A4-5, A4-6, A4-7, A4-8, A4-9, A4-10, A4-11, A4-12, A4-13, A4-14, A5-6, A5-7, A5-8, A5-9, A5-10, A5-11, A5-12, A5-13, A5-14, A6-7, A6-8, A6-10, A6-11, A6-12, A6-13, A6-14, A7-8, A7-9, A7-11, A7-12, A8-10, A8-11, A8-12, A8-13, A8-14, A9-10, A9-11, A9-12, A9-13, A9-14, A10-11, A11-12, A11-13, A11-14, A12-14

- *B*: B1-2, B1-3, B1-4, B1-5, B1-6, B1-7, B1-10, B1-11, B1-12, B1-13, B1-14, B2-3, B2-4, B2-5, B2-6, B2-7, B2-8, B2-9, B2-10, B2-11, B2-12, B2-13, B2-14, B3-4, B3-5, B3-7, B3-8, B3-9, B3-10, B3-11, B3-12, B3-13, B3-14, B4-5, B4-6, B4-7, B4-8, B4-9, B4-10, B4-11, B4-12, B4-13, B4-14, B5-6, B5-7, B5-8, B5-9, B5-10, B5-11, B5-12, B5-13, B5-14, B6-7, B6-8, B6-9, B6-10, B6-11, B6-12, B6-13, B6-14, B7-8, B7-9, B7-10, B7-11, B7-12, B7-13, B7-14, B8-9, B8-10, B8-11, B8-12, B8-13, B8-14, B9-10, B9-11, B9-12, B9-13, B9-14, B10-11, B10-12, B11-13, B11-14, B12-13, B12-14, B13-14

- *C*: C1-2, C1-3, C1-4, C1-5, C1-6, C1-7, C1-8, C1-9, C1-10, C1-11, C1-12, C1-13, C1-14, C2-3, C2-4, C2-5, C2-6, C2-7, C2-8, C2-9, C2-10, C2-11, C2-12, C2-13, C2-14, C3-4, C3-5, C3-6, C3-7, C3-8, C3-9, C3-10, C3-14, C4-5, C4-6, C4-7, C4-9, C4-10, C4-11, C4-12, C4-13, C4-14, C5-6, C5-7, C5-8, C5-10, C5-11, C5-12, C5-13, C5-14, C6-7, C6-8, C6-9, C6-10, C6-11, C6-12, C6-13, C6-14, C7-8, C7-9, C7-10, C7-11, C7-12, C7-13, C7-14, C8-9, C8-10, C8-11, C8-12, C8-13, C8-14, C9-10, C9-11, C9-12, C9-13, C9-14, C10-11, C10-12, C10-13, C11-14, C12-14, C13-14,

- *D*: D1-2, D1-3, D1-4, D1-5, D1-6, D1-7, D1-8, D1-9, D1-10, D1-11, D1-12, D1-13, D1-14, D2-3, D2-4, D2-5, D2-6, D2-7, D2-8, D2-9, D2-10, D2-11, D2-13, D2-14, D3-4, D3-5, D3-6, D3-7, D3-8, D3-9, D3-10, D3-11, D3-12, D3-14, D4-5, D4-6, D4-8, D4-9, D4-10, D4-11, D4-12, D4-13, D4-14, D5-6, D5-7, D5-8, D5-10, D5-12, D5-13, D5-14, D6-7, D6-8, D6-9, D6-10, D6-11, D6-12, D6-13, D6-14, D7-8, D7-9, D7-11, D7-12, D7-13, D7-14, D8-9, D8-10, D8-11, D8-12, D8-13, D8-14, D9-10, D9-12, D9-13, D9-14, D10-11, D10-12, D10-13, D10-14, D11-12, D11-13, D11-14, D12-13, D12-13, D12-14, D13-14

- *E*: E1-2, E1-6, E1-7, E1-10, E1-11, E1-12, E1-13, E1-14, E2-3, E2-4, E2-5, E2-6, E2-7, E2-8, E2-9, E2-10, E2-11, E2-12, E2-13, E2-14, E3-5, E3-6, E3-7, E3-9, E3-10, E3-11, E3-12, E3-13, E3-14, E4-6, E4-7, E4-10, E4-11, E4-12, E4-13, E4-14, E5-6, E5-7, E5-9, E5-10, E5-11, E5-12, E5-13, E5-14, E6-7, E6-8, E6-9, E6-10, E6-11, E6-12, E6-13, E6-14, E7-8, E7-9, E7-10, E7-11, E7-12, E7-13, E7-14, E8-10, E8-11, E8-12, E8-13, E8-14, E9-10, E9-11, E-12, E9-13, E9-14, E10-12, E11-12, E11-13, E12-13, E12-14

- *F*: F1-2, F1-3, F1-6, F1-7, F1-8, F1-9, F1-10, F1-11, F1-12, F1-13, F1-14, F2-4, F2-5, F2-6, F2-7, F2-8, F2-9, F2-10, F2-11, F2-12, F2-13, F2-14, F3-4, F3-5, F3-6, F3-7, F3-8, F3-9, F3-10, F3-11, F3-12, F3-13, F3-14, F4-6, F4-7, F4-8, F4-9, F4-10, F4-11, F4-12, F4-13, F4-14, F5-6, F5-7, F5-8, F5-9, F5-10, F5-11, F5-12, F5-13, F5-14, F6-7, F6-8, F6-9, F6-10, F6-11, F6-12, F6-13, F6-14, F7-8, F7-9, F7-10, F7-11, F7-12, F7-13, F7-14, F8-9, F8-10, F8-11, F8-12, F8-13, F8-14, F9-10, F9-11, F9-12, F9-13, F9-14, F10-12, F10-13, F11-12, F11-13, F12-14, F13-14

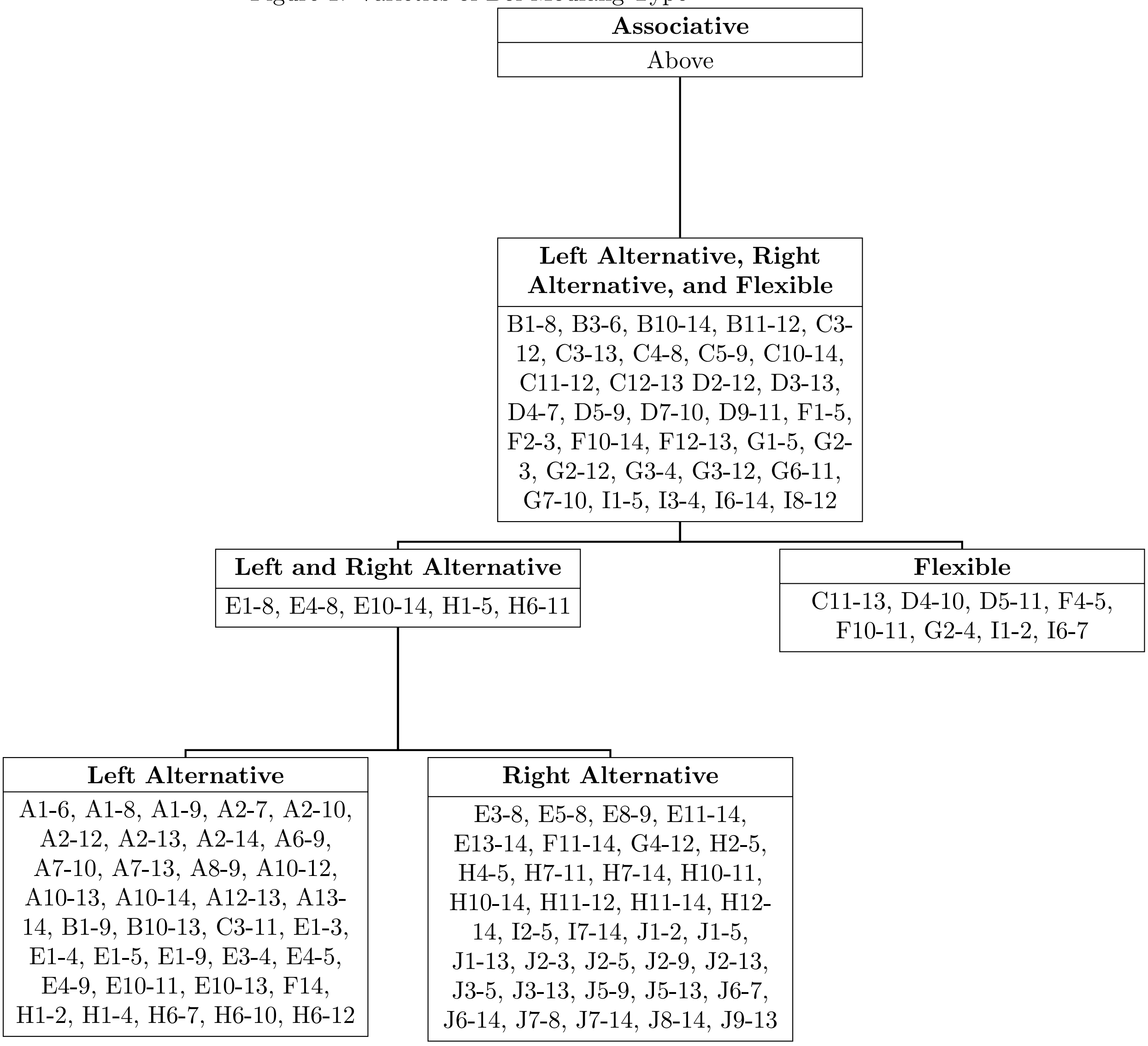
- *G*: G1-2, G1-3, G1-4, G1-6, G1-7, G1-8, G1-9, G1-10, G1-11, G1-12, G1-13, G1-14, G2-5, G2-6, G2-7, G2-8, G2-9, G2-10, G2-11, G2-13, G2-14, G3-5, G3-6, G3-7, G3-8, G3-9, G3-10, G3-11, G3-13, G3-14, G4-5, G4-6, G4-7, G4-8, G4-9, G4-10, G4-11, G4-13, G4-14, G5-6, G5-7, G5-8, G5-9, G5-10, G5-11, G5-12, G5-13, G5-14, G6-7, G6-8, G6-9, G6-10, G6-12, G6-13, G6-14, G7-8, G7-9, G7-11, G7-12, G7-13, G7-14, G8-9, G8-10, G8-11, G8-12, G8-13, G8-14, G9-10, G9-11, G9-12, G9-13, G9-14, G10-11, G10-12, G10-13, G10-14, G11-12, G11-13, G11-14, G12-13, G12-14, G13-14

- *H*: H1-3, H1-6, H1-7, H1-8, H1-9, H1-10, H1-11, H1-12, H1-13, H1-14, H2-3, H2-4, H2-6, H2-7, H2-8, H2-9, H2-10, H2-11, H2-12, H2-13, H2-14, H3-4, H3-5, H3-6, H3-7, H3-8, H3-9, H3-10, H3-11, H3-12, H3-13, H3-14, H4-6, H4-7, H4-8, H4-9, H4-10, H4-11, H4-12, H4-13, H4-14, H5-6, H5-7, H5-8, H5-9, H5-10, H5-11, H5-12, H5-13, H5-14, H6-8, H6-9, H6-13, H6-14, H7-8, H7-9, H7-10, H7-12, H7-13, H8-9, H8-10, H8-11, H8-12, H8-13, H8-14, H9-10, H9-11, H9-12, H9-13, H9-14, H10-12, H10-13, H11-13, H12-13, H13-14

- *I*: I1-3, I1-4, I1-6, I1-7, I1-8, I1-9, I1-10, I1-11, I1-12, I1-13, I1-14, I2-3, I2-4, I2-6, I2-7, I2-8, I2-9, I2-10, I2-11, I2-12, I2-13, I2-14, I3-5, I3-6, I3-7, I3-8, I3-9, I3-10, I3-11, I3-12, I3-13, I3-14, I4-5, I4-6, I4-7, I4-8, I4-9, I4-10, I4-11, I4-12, I4-13, I4-14, I5-6, I5-7, I5-8, I5-9, I5-10, I5-11, I5-12, I5-13, I5-14, I6-8, I6-9, I6-10, I6-11, I6-12, I6-13, I7-8, I7-9, I7-10, I7-11, I7-12, I7-13, I8-9, I8-10, I8-11, I8-13, I8-14, I9-10, I9-11, I9-12, I9-13, I9-14, I10-11, I10-12, I10-13, I10-14, I11-12, I11-13, I11-14, I12-13, I12-14, I13-14

- *J*: J1-3, J1-4, J1-6, J1-7, J1-8, J1-10, J1-11, J1-12, J1-14, J2-4, J2-6, J2-7, J2-8, J2-10, J2-11, J2-12, J2-14, J3-4, J3-6, J3-7, J3-8, J3-9, J3-10, J3-11, J3-12, J3-14, J4-5, J4-6, J4-7, J4-8, J4-9, J4-10, J4-11, J4-12, J4-13, J4-14,J5-6, J5-7, J5-8, J5-10, J5-11, J5-12, J5-14, J6-8, J6-9, J6-10, J6-11, J6-12, J6-13, J7-9, J7-10, J7-11, J7-12, J7-13, J8-9, J8-10, J8-11, J8-12, J8-13, J9-10, J9-11, J9-12, J9-14, J10-11, J10-12, J10-13, J10-14, J11-12, J11-13, J11-14, J12-13, J12-14, J13-14

Figure 1: Varieties of Bol-Moufang Type



The diagram is set up so that the identities in each box imply the property or properties listed at the top of the box. Therefore, an identity in one box will imply any of the identities that are contained in boxes below it, but the implications do not necessarily go the opposite direction in the chart.

Many of the above implications were found using Prover9. Thus, we found it useful to include select proofs of some of the implications.

- $C10-14 \Rightarrow D5-11:$

  Proof. C10-14: $(x(y(zx)))u = (((xy)z)x)u$. Since we are in a loop, there exists an identity element e such that $xe = x$ and $ex = x$ for all x in the loop.

  We need to show D5-11 holds true, i.e. $x(((yz)u)x) = (x((yz)u))x$.

  Consider C10-14. Let $z = e$. Then $((xy)x)u = (x(yx))u$.

  Now we are able to left cancel u, so $(xy)x = x(yx)$.

  Let $y = (yz)u$. Then $(x((yz)u))x = x(((yz)u)x$, which is D5-11, as desired.

  □

- $H6-11 \Rightarrow J1-2:$

  Proof. H6-11: $(xy)(z(zu)) = (x((yz)z))u$. Since we are in a loop, there exists an identity element e such that $xe = x$ and $ex = x$ for all x in the loop.

  We need to show J1-2 holds true, i.e. $x(y(z(uu))) = x(y((zu)u))$.

  Consider H6-11. Let $u = e$. Then $(xy)(zz) = x((yz)z)$.

  Now let $x = e$. Then $y(zz) = (yz)z$.

  Without loss of generality, we can consider $z(uu) = (zu)u$. Now left multiply by y. Then left multiply again by x. We obtain $x(y(z(uu))) = x(y((zu)u))$, which is J1-2, as desired.

  □

- $E1-3 \Rightarrow \ A1-6$

  Proof. We start by assuming that the following equations hold true:

$$x[y(y(zu))] = [(xy)y](zu) \tag{23}$$

$$(x/y)y = x \tag{24}$$

$$ex = x \tag{25}$$

  Making the substitutions $(x, y, z, u) \mapsto (x, y, z/u, u)$ into (23) gives us

$$x[y(y((z/u)u))] = [(xy)y][(z/u)u] \tag{26}$$

  Using (24) we get

$$x[y(yz)] = [(xy)y]z \tag{27}$$

  Making the substitutions $(x, y, z) \mapsto (e, x, y)$ into (27) gives us

$$e[x(xy)] = [(ex)x]y \tag{28}$$

Using (25) we get

$$x(xy) = (xx)y \tag{29}$$

Making the substitutions $(x, y) \mapsto (x, y(zu))$ into (29) gives us

$$x[x(y(zu))] = (xx)[y(zu)] \tag{30}$$

This is $A1 - 6$, which is what we were trying to prove.

□

4. Future Research

Future research could easily be adapted from the work presented in this paper. One could examine identities with more variables and parenthesization patterns. It would also be possible to classify the five variable case (four distinct with one repeated) in a quasigroup instead of a loop.

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6. References

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