The Structure of Zero Divisor Sum Graphs

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Abstract

Let $\Sigma_n$ be the graph whose vertex set is the set of non-zero zero divisors of $\mathbb{Z}_n$ where $vw$ is an edge if $v+w$ is a non-zero zero divisor. We study various graph-theoretic properties of $\Sigma_n$, including vertex degree, connectivity and cycles. Further investigation is also made into planar graphs and automorphisms of these kinds of graphs.

1 Preliminaries

1.1 Ring Structure

In order to fully understand the topic at hand it is important to review the properties of rings. A ring is a non-empty set $R$ that has two binary operations, addition and multiplication, satisfying the following:

- $(R, +)$ is an abelian group.
- Multiplication is associative and commutative.
- For all $a, b, c \in R$, the distributive law, $a(b + c) = (ab) + (ac)$ holds.

We use the ring $\mathbb{Z}_n$, this is the set of integers $0, 1, \ldots, (n - 1)$ under addition and multiplication modulo $n$. One of the most important concepts is that of a zero divisor.

Definition 1.1. In a ring $R$, a zero divisor is an element $z \in R$ such that there exists $x \in R$, $x \neq 0$ where $zx = 0$.

Theorem 1.2. [4] In a ring $\mathbb{Z}_n$, the zero divisors are precisely those non-zero elements that are not relatively prime to $n$.

In other words, $k$ is a zero divisor in $\mathbb{Z}_n$ if and only if $\gcd(k, n) > 1$, a fact which we will use often throughout the paper. The other important class of elements are the units.
Definition 1.3. A unit is an element $u \in R$ that has a multiplicative inverse in $R$, i.e. there exists a $v \in R$ such that $uv = 1$.

Theorem 1.4. [4] Let $n > 0$. Every element of $\mathbb{Z}_n$ is either a zero divisor or a unit.

1.2 Graph Structure

We use a graph theoretic approach to study the non-zero zero divisors of the ring $\mathbb{Z}_n$.

Definition 1.5. A graph $G$ consists of a vertex set, $V(G)$, an edge set $E(G)$, and an association to each edge, $e \in E(G)$ of two vertices called the endpoints of $e$.

Two vertices are adjacent if they share a common edge. If $x$ and $y$ are adjacent in a graph we denotes this $x \sim y$.

Two adjacent vertices are referred to as neighbors of each other. The set of neighbors of a vertex $v$, denoted $N(v)$ is called the neighborhood of $v$. An edge is incident at a vertex if that vertex is one of its endpoints. The degree of a vertex $v$ is the number of edges incident at it, denoted by $\deg(v)$.

A $uv$-walk in a graph $G$ is a sequence of vertices in $G$ beginning with $u$ and ending at $v$ such that consecutive vertices in the sequence are adjacent. A trail in a graph $G$ is a walk in which no edge is traversed more than once. A $uv$-path is a walk in a graph starting at a vertex $u$ and ending at a vertex $v$ in which no vertex is repeated except for maybe the first and the last. A cycle is a closed path.

Definition 1.6. Let $G$ be a graph. A subgraph of $G$ is a graph $H$ such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

1.3 Sum Graphs

Definition 1.7. A sum graph is a graph whose vertices are labeled with integers where vertices $i$ and $j$ are joined by a line if and only if the vertex $i + j$ is in the graph.

We investigate the graph $\Sigma_n$. The vertex set $V(\Sigma_n)$ consists of the non-zero zero divisors of $\mathbb{Z}_n$. Distinct vertices $x, y \in V(\Sigma_n)$ are adjacent if $x + y \in V(\Sigma_n)$.

It is important to notice that no vertex $v$ is adjacent to its inverse $-v$ because this would imply that 0 is in the graph.
Note that when \( n \) is prime, \( V(\Sigma_n) = \emptyset \). For our study we will never consider this case.

**Theorem 1.8.** If \( h \mid n \), then \( \Sigma_h \) is a subgraph of \( \Sigma_n \).

**Proof.**
For any \( k \in V(\Sigma_h) \), the gcd\((h, k) > 1\). Since \( h \mid n \), then gcd\((n, k) > 1\). Thus, every vertex in \( \Sigma_h \) is a vertex in \( \Sigma_n \).

For all edges \( xy \) in \( \Sigma_h \), we know that \( x, y, x + y \in V(\Sigma_h) \). From the statement above, we know that \( x, y, x + y \in V(\Sigma_n) \). Therefore, \( \Sigma_h \) is a subgraph of \( \Sigma_n \). ☐

## 2 Vertex Degrees

**Definition 2.1.** A dominating vertex is a vertex in a graph that is adjacent to every other vertex of \( G \).

**Proposition 2.2.** Let \( n = 2x \) for some integer \( x \). Then \( x \) is a dominating vertex in \( \Sigma_n \) if and only if \( x \) is even.

**Proof.**

(\( \Leftarrow \)) Let \( x \in V(\Sigma_n) \) be even. We want to show for all \( v \in V(\Sigma_n) \), with \( v \neq x \), we have \( v + x \in V(\Sigma_n) \).

Case I: Suppose \( v \) is even, or that \( 2 \mid v \).
Since \( 2 \mid x \), then \( 2 \mid v + x \). This implies that gcd\((v + x, n) \geq 2 > 1\), therefore \( v + x \in V(\Sigma_n) \).

Case II: Suppose \( v \) is odd, or that \( 2 \nmid v \).
Since \( v \) is a zero-divisor, there exists \( w \) such that \( vw = 0 \in \mathbb{Z}_n \), or, viewing \( v \) and \( w \) as integers, \( n \mid vw \). Since \( 2 \mid n \) and \( 2 \nmid v \), we know \( 2 \mid w \). Then
\[
\begin{align*}
w(v + x) &= vw + wx \\
&= vw + w(n/2) \\
&= 0 + (w/2)n \\
&= 0
\end{align*}
\]
as we are in \( \mathbb{Z}_n \) and \( w/2 \) is an integer.

Therefore, for all \( v \in V(\Sigma_n) \), where \( v \neq x \), it follows that \( x \sim v \). Hence, \( x \) is a dominating vertex in \( \Sigma_n \).
(⇒) If $x$ is a dominating vertex, then $x$ is even.

Suppose towards a contradiction that $x$ is odd. Because $x$ is a dominating vertex, $x \sim 2$, so $x + 2 \in V(\Sigma_n)$. Thus $x + 2$ is odd and $\text{gcd}(x + 2, n) > 1$. This forces $x + 2 = x$, since $x$ is the only odd non-zero zero divisor. This is a contradiction because $x + 2 \neq x$, or $2 \neq 0$. Therefore $x$ is even. Hence, $x$ is a dominating vertex if and only if $x$ is even.

\section*{Definition 2.3.}
Let $R$ be a ring, and $a, b \in R$. We say that $a$ is associate to $b$ if $a = ub$ for some unit $u$.

\section*{Proposition 2.4.}
If $g, g' \in V(\Sigma_n)$ and $g$ is associate to $g'$, then

$$\text{deg}(g) = \text{deg}(g').$$

\section*{Proof.}
We wish to show that if $g \sim x$ then there exists $y \in V(\Sigma_n)$ such that $g' \sim y$.

Suppose $g \sim x$. This implies that $g + x$ is a non-zero zero divisor. Since $g$ is associate to $g'$, we know $gk = g'$ for some unit $k$.

Let $y = xk$. Observe that $x$ is associate to $y$. Since $x$ is a zero divisor, there exists non-zero $s \in \mathbb{Z}_n$ such that $xs = 0$. Also observe:

$$ys = (xk)s = (xs)k = 0$$

So $y$ is a zero divisor. We now show that $y$ is a non-zero zero divisor.

Assume that $y = 0$. Since $y = xk$, this implies that $xk = 0$. Since $k$ is a unit, we can multiply by its inverse and end up with $x = 0$. This is a contradiction since $x$ is a non-zero zero divisor. Therefore $y \neq 0$, and $y \in V(\Sigma_n)$.

Now, we show that $g' \sim y$, or that $g' + y$ is a non-zero zero divisor. Observe:

$$g' + y = gk + xk = (g + x)k,$$

and since $g + x$ is a divisor, there exists a non-zero $c \in \mathbb{Z}_n$ such that

$$(g + x)c = 0.$$
and so
\[
((g + x)k)c = 0
\]
Since \(kc \neq 0\), this implies that \(g' + y\) is a non-zero zero divisor and therefore that \(g' \sim y\).

Define the map \(\Phi: N(g) \mapsto N(g')\) by
\[
\Phi(x) = xk.
\]
This map is injective because if \(xk = yk\), multiplying on the right side by \(k^{-1}\) gives
\[
x = y.
\]
Thus, for each neighbor of \(g\) we are able to find a unique neighbor of \(g'\). Therefore, \(\deg(g') \geq \deg(g)\). By symmetry, \(\deg(g) \geq \deg(g')\). Therefore, if \(g\) is associate to \(g'\),
\[
\deg(g) = \deg(g').
\]

\[\Box\]

**Definition 2.5.** The minimum degree of a graph \(G\) is the smallest degree of all the vertices in a graph, denoted \(\delta(G)\).

**Proposition 2.6.** \(\Sigma_n, \delta(\Sigma_n) = 1\) if and only if \(n = 4q\), where \(q\) is prime.

**Proof.**

(\(\Leftarrow\)) Notice \(V(\Sigma_n) = A \cup B\) where \(A = \{2, 4, 6, \ldots, 4q - 2\}\) and \(B = \{q, 2q, 3q\}\).

Each element in \(A\) is adjacent to every other element in \(A\) except its inverse. Thus \(\deg(v) > 1\) for all \(v \in A\). Also \(q \sim 2q\), but \(q \not\sim 3q\)(because they are inverses) and \(q \not\sim v\) for all \(v \in A \setminus \{2q\}\) because the sum of an even number and an odd number is odd. Therefore \(q\) will have degree 1.

(\(\Rightarrow\)) Let \(v \in V(\Sigma_n)\) such that \(\deg(v) = 1\).

**Case I:** \(n = tv\) where \(t \geq 5\).

Then, \(v, 2v, 3v, 4v \in V(\Sigma_n)\). As a result, \(v \sim 2v\) and \(v \sim 3v\). So, \(\deg(v) > 1\), which is a contradiction.

**Case II:** \(n = 2v\)

This implies \(v = -v\). Since \(\deg(v) = 1\), \(v\) is adjacent to some vertex, say \(x\). This means
\[
\begin{align*}
v + x &\in V(\Sigma_n) \\
\Rightarrow -v - x &\in V(\Sigma_n) \\
\Rightarrow v - x &\in V(\Sigma_n) \text{ since } v = -v \\
\Rightarrow v &\sim -x
\end{align*}
\]
Suppose \( x = -x \). Then, since \( x < n \), we have \( n = 2x \). Thus \( v = \frac{n}{2} = x \). This is a contradiction because \( v \) is not adjacent to itself. So \( x \neq -x \) which implies \( \deg(v) > 1 \), a contradiction. Therefore \( n \neq 2v \).

Case III: \( n = 3v \)
Subcase A: If \( 2 \mid n \), and since \( 3 \mid n \), this implies that \( 6 \mid n \). Since \( 6 \mid n \), this subcase has been reduced to Case 1.

Subcase B: If \( 2 \nmid n \), then 2 is a unit in \( \mathbb{Z}_n \). Now \( -2v = v \) (as \( 3v = 0 \)). Since \( \deg(v) = 1 \), suppose \( v \sim x \). This implies

\[
\begin{align*}
v + x & \in V(\Sigma_n) \\
\Rightarrow (-2)(v + x) & \in V(\Sigma_n) \text{ [as } -1 \text{ and } 2 \text{ are units.]} \\
\Rightarrow v - 2x & \in V(\Sigma_n) \\
\Rightarrow v & \sim -2x
\end{align*}
\]

Suppose \( x = -2x \). Then, since \( x < n \), we have \( x = \frac{n}{3} \) or \( x = \frac{2n}{3} \). Thus either \( v = x \) or \( v = -x \). This is a contradiction because \( v \) is not adjacent to itself or to its inverse. So \( x \neq -2x \) which implies \( \deg(v) > 1 \), a contradiction. Therefore \( n \neq 3v \).

Thus we have shown that the minimum degree of \( \Sigma_n \) is 1 when \( n = 4v \). However, we must still show \( v \) is prime. Suppose towards contradiction \( v = st \) where \( s, t > 1 \). Then \( n = 4st \). We see

\[
v \sim s \text{ as } s(t + 1) \text{ is a zero divisor and } st + s \neq n \text{ and } v \sim 2s \text{ as } s(t + 2) \text{ is a zero divisor and } st + 2s \neq n.
\]

Thus \( \deg(v) > 1 \), which is a contradiction. So \( v \) is prime.

\[\Box\]

**Proposition 2.7.** There exists an isolated vertex, \( \delta(\Sigma_n) = 0 \), if and only if \( n = 3p \) or \( n = 2p \), where \( p \) is prime.

**Proof.** We will consider the two cases, \( n = 2p \) or \( n = 3p \).

(\( \Leftarrow \)) Assume \( n = 3p \).

The vertex set of \( \Sigma_{3p} \) is of the form: \( A \cup B \) where

\[
A = \{3, 6, \ldots, 3p - 3\}
\]

and

\[
B = \{p, 2p\}
\]

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Notice, \( p \not\sim 2p \) because \( p + 2p \not\in V(\Sigma_{3p}) \).

Let \( a \in A \). For all these vertices, \( p \not\sim a \) because the vertex set \( A \) contains only multiples of 3. And \( p \) is not a multiple of 3. So \( p + a \not\in A \) and \( 2p + a \not\in A \). Also, for every \( b \in B \) we have \( p + b \not= 2p \) and \( 2p + b \not= p \), because \( b \) is not a multiple of \( p \). Lastly, \( p \not\sim 2p \) because \( p + 2p = 3p \), and \( 3p \not\in V(\Sigma_{3p}) \).

Therefore \( p \) and \( 2p \) are isolated vertices in \( \Sigma_{3p} \).

Note: If \( p=3 \), then \( n=9 \), which consists of two vertices, which are isolated.

\((\Leftarrow)\) Assume \( n=2p \).
Consider \( \Sigma_n \) where \( n = 2p \) and \( p \) is prime. In this case, the vertices of \( \Sigma_n \) are of the form:

\[ V = \{p, 2, 4, 6, ..., 2p - 2\} \]

Suppose towards contradiction that \( x \sim p \); therefore \( x + p \in V(\Sigma_n) \). The vertex \( x \) corresponds to an even integer.
Because the sum of an even and an odd number is odd, and \( p \) is the only odd vertex,

\[ x + p \equiv p \pmod{2p} \]

This implies that \( x = 0 \). However, the element 0 is not in the graph and therefore we have a contradiction. Hence, \( p \) is an isolated vertex of the graph.

Note: If \( p=2 \), then \( n=4 \), which is an isolated vertex.

\((\Rightarrow)\) Case I: Suppose \( n = tp \), where \( t \geq 4 \) and \( t \) is not prime.

Let \( A = \{p, 2p, ..., (t-1)p\} \). Since \( t \geq 4 \), there are at least three elements in \( A \). Let \( r \) be such that \( r \mid t \) and let \( B = \{r, 2r, ..., (p-1)r\} \). Since there are at least three elements in \( r \), \( \deg (b) > 0 \) for \( b \in B \). There are at least \( p - 1 \) elements in \( B \). If \( p \geq 5 \), there are at least 3 elements in the set, and thus we satisfy our conditions. If \( p < 5 \), consider the following two cases.

Case II: Let \( p = 2 \) so that \( n = 2t \), where \( t \) is not prime.
Since \( t \) is not prime, it can be said that \( t = mp \), where \( p \) is prime and \( m \geq 2 \). Then
by substitution, \( n = 2mp \). Since \( m \geq 2 \), this implies that \( 2m \geq 4 \), which was treated in Case 1.

Case III: Let \( p = 3 \) so that \( n = 3t \), where \( t \) is not prime.
Since \( t \) is not prime, it can be said that \( t = mp \), where \( p \) is prime and \( m > 1 \). Then by substitution, \( n = 3mp \). Since \( m > 1 \), this implies that \( 3m \geq 3 \), which was treated in Case 1.

Case IV: Suppose \( n = tp \) where \( t \) is prime, where \( t, p > 3 \).
Notice \( V(\sigma(tp)) \) can be partitioned into two sets:

\[
A = \{p, 2p, \ldots, (q-1)p\}
\]

\[
B = \{q, 2q, \ldots, (p-1)q\}
\]

All the elements of \( A \) are adjacent to everything except for itself and its inverse. All the elements of \( B \) are adjacent to everything except for itself and its inverse. Since \( t \) and \( p \) are primes greater than three, the degree of any element in \( A \) and \( B \) will be greater than zero.

**Proposition 2.8.** Let \( p > 3 \) be prime. Then the graph \( \Sigma_{p^k} \) for \( k \in \mathbb{Z} \), has \( p^{k-1} - 1 \) vertices, each of degree \( p^{k-1} - 3 \). In particular, all vertices have even degree.

**Proof.**
Observe that

\[
V(\Sigma_{p^k}) = \{p, 2p, 3p, \ldots, (p-1)p, p^2, \ldots, (p^{k-1} - 1)p\}.
\]

In particular there are \( p^{k-1} - 1 \) vertices in the graph. Each vertex is adjacent to all others except itself and its inverse in \( \mathbb{Z}_{p^k} \).

We show that no vertex can be its own inverse. If \( x \in V(\Sigma_{p^k}) \) and \( x = -x \), then \( 2x \equiv 0 \pmod{p^k} \). Because \( p > 3 \) is prime, \( \gcd(2, p^k) = 1 \). Since 2 is a unit in \( \mathbb{Z}_{p^k} \), we can multiply by the inverse of 2, which yields \( x \equiv 0 \pmod{p^k} \), a contradiction.

Since there are \( p^{k-1} - 3 \) neighbors of each vertex, the degree of each vertex in \( \Sigma_{p^k} \) is \( p^{k-1} - 3 \).  

\[\blacksquare\]
3 Connectedness

Recall that $uv$-path is a walk in a graph from a vertex $u$ to a vertex $v$ in which no vertices are repeated.

**Definition 3.1.** A graph $G$ is connected if there exists a $uv$-path between all pairs of distinct vertices $u$ and $v$ of $G$.

**Definition 3.2.** The distance between $u, v \in V(G)$ is the smallest length of a $uv$-path in $G$ and is denoted $d(u, v)$.

**Definition 3.3.** The diameter of a graph is the greatest distance between any two vertices of a connected graph $G$ and is denoted $\text{diam}(G)$.

**Theorem 3.4.** The graph $\Sigma_n$ is disconnected if and only if:

\[
\begin{align*}
&n = 9 \\
&n = pq \text{ where } p \text{ and } q \text{ are distinct primes.}
\end{align*}
\]

**Proof.**

$(\Leftarrow)$ The graph $\Sigma_9$ has two vertices and no edges, hence $\Sigma_9$ is disconnected.

Now let $n = pq$, where $p$ and $q$ are distinct primes. We can partition the vertices into two sets:

\[
A = \{p, 2p, ..., (q-1)p\} \\
B = \{q, 2q, ..., (p-1)q\}
\]

There are no edges between vertices in $A$ and vertices in $B$ because no vertex that is a multiple of $q$ is adjacent to a vertex that is a multiple of $p$. Therefore $\Sigma_n$ has at two connected components and so it is disconnected.

In the case where $n = 6$, there are three isolated vertices and therefore disconnected.

$(\Rightarrow)$ We now show that for every $n$ not of this form, $\Sigma_n$ is connected. So we considered all of the possible prime factorizations of $n$ that are not of the form $3^2$ or $pq$.

Case I: Let $n = p^k$ where $k > 1$ and $n \neq 9$. In order to show that $\Sigma_n$ is connected, it suffices to show that there exists a $uv$-path between all vertices $u$ and $v$ in $\Sigma_n$. 


Let $u \in V(\Sigma_{pq})$. We know from Proposition 2.8 that the graph of $\Sigma_{pq}$ has $p^{k-1} - 1$ vertices each of degree $p^{k-1} - 3$. For distinct $u, v \in V(\Sigma_{pq})$, if $u \not\sim v$, then $u$ and $v$ are inverses. Therefore, there must be another vertex $r$ such that $u \sim r$ and $v \sim r$. Since inverses are unique, $r$ cannot be the inverse of either $u$ or $v$. Therefore there exists an $uv$-path through $r$. Hence, $\Sigma_{pq}$ is connected with diameter 2.

Case II: Let $n = p^aq^b$ where $p, q$ are distinct primes and $a > 1$ or $b > 1$. Without loss of generality, assume that $a > 1$.

In this case, $pq \not\sim pq$ and $pq \not\sim (n - pq)$. However, $pq$ is adjacent to every other vertex in the graph; therefore $pq \sim p$. We know that $p \sim (n - pq)$, so $pq, p, (n - pq)$ is a path. Therefore the graph is connected and has diameter 2.

Case III: Let $n = p_1^{e_1}p_2^{e_2}...p_r^{e_r}$ for $r \geq 3$ where the $p_i$ are distinct primes. Let

$$a_i = n/p_i = p_1^{e_1}p_2^{e_2}...p_i^{e_i-1}...p_r^{e_r}$$

For any $i, j \in \mathbb{Z}$, $a_i \sim a_j$ because all $a_i$ are multiples of $p_k$, and therefore $a_i + a_j$ is a multiple of $p_k$ where $k \neq i, j$ and $k \in \mathbb{N}$. The vertex $a_j$ is of the form:

$$a_j = p_1^{e_1}p_2^{e_2}...p_j^{e_j-1}...p_r^{e_r}.$$ 

The inverse of $a_i$ is $-a_i = p_1^{e_1}p_2^{e_2}... (p_i^{e_i} - p_i^{e_i-1})...p_r^{e_r}$, so $a_j \neq -a_i$, and since $a_j$ is a multiple of $p_k$, it is true that $a_i \sim a_j$ for all $i, j$. So, $a_1, a_2, ..., a_r$ form a clique and are therefore in the same component.

Now we want to show that for all $v \in V(\Sigma_n)$, $v \sim a_s$ for all $s$ except maybe one. Suppose $v = -a_i$ for some $i$. Then obviously, $v + a_j \neq 0$ for all $j \neq i$. So we have shown that $v + a_j$ is non-zero.

Now we need to show that $\gcd(n, v) > 1$. We know that $v$ is a multiple of $p_k$ for all $k \neq i$, and that $a_j$ is a multiple of all $p_k$ for $k \neq j$. Since $r \geq 3$ we know we can pick $k \neq i, j$. Therefore $p_k$ is a factor of $a_j$ and $v$ and since $a_j$ and $v$ are multiples of $p_k$, it is true that $a_j + v$ is a multiple of $p_k$ and hence, a zero divisor. So, $v \sim a_k$ for all $k \neq i$ if $v = -a_i$.

Now, suppose $v$ is not of the form $a_i$ for any $i$. Then there exists $i$ such that $\gcd(v, p_i) > 1$, therefore $v$ is a multiple of $p_i$. For all $s \neq i$, we know $a_s$ is a multiple
of $p_i$, since $v$ is a multiple of $p_i$ and $a_s$ is a multiple of $p_i$, it is true that $v + a_s$ is a multiple of $p_i$ and therefore is a zero divisor. Here, their sum is obviously not 0 because $v$ is not the inverse of any $a_i$.

Therefore $v \sim a_s$ for all $s \neq i$.

Finally, we need to show that for $u, v \in V(\Sigma_n)$, there exists a $uv$-path. Find $i, j \in V(\Sigma_n)$ such that $\gcd(p_i, u) > 1$ and $\gcd(p_j, v) > 1$. Since $r > 3$, we find $k \neq i, j$, therefore $u \sim a_k$ and $v \sim a_k$ and thus there exists a $uv$-path of distance 2, so $\Sigma_n$ is connected.

Therefore, a $\Sigma_n$ is disconnected if and only if $n = 9$ or $n = pq$. ▲

Corollary 3.5. If $\Sigma_n$ is connected, its diameter is less than or equal to 2.

Proof. From the proof of Proposition 3.4 we can see that $\text{diam}(\Sigma_n) \leq 2$ when $\Sigma_n$ is connected. ▲

Definition 3.6. A $uv$-trail in a graph is a $uv$-walk in which no edge is traversed more than once.

Definition 3.7. An Eulerian graph is a graph which contains a closed trail containing every edge.

Theorem 3.8. [3] A nontrivial connected graph is Eulerian if and only if every vertex of $G$ has even degree.

Combining Propositions 2.8 and Case I of 3.4, we develop the following corollary.

Corollary 3.9. Let $p > 3$ be prime, then $\Sigma_{p^k}$ is Eulerian.

Proof.
From Proposition 2.8 we see that the vertices of the graph of $\Sigma_{p^k}$ are all of even degree, and from Proposition 3.4 we see that $\Sigma_{p^k}$ is connected. Therefore by Theorem 3.8, $\Sigma_{p^k}$ is Eulerian. ▲
4 Cycles

Theorem 4.1. Let \( n \) be composite, then \( \Sigma_n \) contains a cycle if and only if \( n > 9 \).

Proof.

\((\Leftarrow)\) Let \( n > 9 \) be such that \( n = pm \) where \( p \) is the smallest prime that divides \( n \) and \( m \in \mathbb{N} \). Note that this implies \( m \geq 4 \).

If \( n = 4p \), then \( p \neq 2 \) because \( n > 9 \). However, if \( p \geq 3 \), then \( 2, 4, 6 \) form a cycle.

If \( n = 5p \) then \( 2p \not\sim 3p \), but \( p, 2p, 4p, 3p \) form a cycle.

If \( n \) is not of the form \( 4p \) or \( 5p \), then \( p, 2p, 3p \) form a cycle.

\((\Rightarrow)\) Now we show by brute force that for \( n \leq 9 \) the graph of \( \Sigma_n \) does not contain a cycle.

For \( n = 1, 2, 3, 5, 7, \) \( n \) is prime, so \( \Sigma_n \) obviously contains no cycle because it is the empty set.

For \( n = 4 \), the graph contains only the vertex 2, so \( \Sigma_4 \) does not contain a cycle either.

For \( n = 6 \), the graph contains 3 vertices, but 3 is an isolated vertex, so there is no cycle.

For \( n = 8 \), the graph contains 3 vertices, but 2 and 6 are not adjacent, so there is no cycle.

Therefore, for composite \( n \), the graph \( \Sigma_n \) contains a cycle if and only if \( n > 9 \). ▲

Definition 4.2. If a graph \( G \) has a cycle, then the girth of \( G \) is the length of the shortest cycle in \( G \). The girth of a graph \( G \) is denoted \( gr(G) \).

Corollary 4.3. If \( n \neq 5p \) and \( n > 9 \) is composite, then \( gr(\Sigma_n) = 3 \).

Proof. This is clear from the proof of Proposition 4.1. ▲

Corollary 4.4. If \( n = 5p \) for \( p = 2 \) or 3, then \( gr(\Sigma_n) = 4 \).

Proof.

Recall that \( p \) is the smallest prime that divides \( n \) where \( n > 9 \). Since \( 5 \mid n \), and \( p \leq 5 \), there are only 3 cases to consider, \( n = 10, n = 15 \), and \( n = 25 \). In each case, inspection shows that \( \Sigma_n \) contains no 3-cycles.
If \( n = 10 \) a cycle is formed by 2, 4, 8, 6 and \( gr(\Sigma_{10}) = 4 \).

Figure 1: This is a picture of \( \Sigma_{10} \); note that there are no three cycles.

If \( n = 15 \) a cycle is formed by 3, 6, 12, 9 and \( gr(\Sigma_{15}) = 4 \).

Figure 2: This is a picture of \( \Sigma_{15} \); note that there are no three cycles.

If \( n = 25 \) a cycle is formed by 5, 10, 20, 15 and again \( gr(\Sigma_{25}) = 4 \).

Figure 3: This is a picture of \( \Sigma_{25} \); note that there are no three cycles.

Thus, \( gr(\Sigma_n) = 4 \) in all 3 cases.
5 Planarity

Definition 5.1. A graph is planar if it can be drawn in the plane with no edge crossings.

To prove our next lemma, we use the Handshaking Theorem[5]:

Theorem 5.2. The sum of the degrees of the vertices of a graph is equal to twice the number of edges.

We denote the number of vertices as $m$ and the number of edges as $e$ to obtain the equation

$$\sum_{v \in V} \deg(v) = 2e.$$

Lemma 5.3. For all graphs, $\delta m/2 \leq e$.

Proof. For every $v \in V(\Sigma_n)$, $\delta \leq \deg(v)$. Therefore, $\delta m \leq \sum_{v \in V} \deg(v)$. By the handshaking theorem, $\delta m \leq 2e$. Dividing by 2, $\delta m/2 \leq e$. ▲

Definition 5.4. A face is a portion of a plane drawing of a planar graph which is bounded by edges and has no edge running through the interior.

Theorem 5.5. [5] (Euler’s Formula) For any connected plane graph $G$, $m - e + f = 2$, where $f$ denotes the number of faces in $G$.

Lemma 5.6. If $\Sigma_n$ is planar, then $e \leq 3m - 6$.

Proof. Every face in $\Sigma_n$ has length $\geq 3$. Let $F_1, F_2, \ldots, F_k$ be the faces in $\Sigma_n$ and $l(F_i)$ be the length of the boundary of the face $F_i$. Because every edge $e$ is part of the boundary of two faces, $2e = \sum_1^k l(F_i) \geq \sum_1^k 3 = 3f$. So $f \leq 2e/3$. Using Euler’s formula, we know

$$f = 2 - m + e \leq 2e/3$$
$$e/3 \leq m - 2$$
$$e \leq 3m - 6$$

▲

We use Lemma 5.3 and Lemma 5.6 to obtain the following inequalities:
Theorem 5.7. Let \( p \) be prime and \( k > 1 \). If \( \Sigma_{p^k} \) is planar, then \( k \leq 3 \). Further, \( p \leq 7 \) for \( k \leq 2 \) and \( p = 2 \) for \( k = 3 \).

Proof.

By the result above, \( \delta \leq 6 - 12/m \). Let \( i \) be any integer such that \( 1 < i \leq k \) and let \( c \) be any integer such that \( 1 \leq c \leq p - 1 \). Since \( p \) is prime, \( cp^i \in V(\Sigma_{p^k}) \) is adjacent to every vertex in \( V(\Sigma_{p^k}) \) except for itself and \( p^k - cp^i \). Therefore, \( \delta = m - 2 \), and we have

\[
\begin{align*}
    m - 2 & \leq 6 - 12/m \\
    m - 8 + 12/m & \leq 0 \\
    m^2 - 8m + 12 & \leq 0 \\
    (m - 6)(m - 2) & \leq 0 \\
    m & \leq 6
\end{align*}
\]

But we know that \( m = p^{k-1} - 1 \), so

\[
\begin{align*}
    m = p^{k-1} - 1 & \leq 6 \\
    p^{k-1} & \leq 7
\end{align*}
\]

If \( k = 2 \), then \( p \leq 7 \).

If \( k = 3 \), then \( p^2 \leq 7 \), and \( p = 2 \).

If \( k \geq 4 \), then \( p^{k-1} \leq 7 \), and \( p < 2 \), which is impossible. \( \blacksquare \)

Definition 5.8. A subdivision of an edge \( e = uv \) occurs when a new vertex \( w \) is placed along \( e \) and the edge \( uv \) is replaced by the path \( uwv \) of length 2.

Definition 5.9. Let \( G \) and \( G' \) be graphs. \( G \) is homeomorphic to \( G' \) if there exists a graph \( H \) such that \( G \) and \( G' \) both result from subdivisions of \( E(H) \).

Definition 5.10. The graph \( K_{3,3} \) has six vertices. Three of the vertices \( a, b, \) and \( c \) are all adjacent to the other three vertices \( d, e, \) and \( f \). Also, \( a \sim b, c; b \sim c; d \sim e, f; e \sim f \).

Definition 5.11. The graph \( K_5 \) has five vertices, each of which are adjacent to the other four.
Theorem 5.12. [5] (Kuratowski’s Theorem) A graph is planar if and only if it has no subgraph homeomorphic to $K_5$ or $K_{3,3}$.

Lemma 5.13. Every subgraph of a planar graph is planar.

Proof. Suppose not. Let $G$ be a planar graph and $H$ be a non-planar subgraph of $G$. Then $H$ contains a subgraph homeomorphic to $K_5$ or $K_{3,3}$. So $G$ itself must contain a subgraph that is homeomorphic to $K_5$ or $K_{3,3}$. ▲

Definition 5.14. Let $G$ be a graph and $S \subseteq V(G)$. The subgraph induced by $S$, denoted $G[S]$ is the graph defined by $V(G[S]) = S$ and $e \in E(G[S])$ if $e \in E(G)$ and both endpoints of $e$ are elements of $S$.

Theorem 5.15. Let $p_1^{e_1}, p_2^{e_2}, \ldots, p_r^{e_r}$ be primes. If $\Sigma_{p_1^{e_1}, p_2^{e_2}\ldots p_r^{e_r}}$ is planar, then $p_i \leq 7$ for every $i$.

Proof. Let $I$ be the subgraph of $\Sigma_{p_1^{e_1}, p_2^{e_2}\ldots p_r^{e_r}}$ induced by $p_i$, $2p_i, \ldots, (p_j-1)p_i$ for some $1 \leq i \leq r$ and $1 \leq j \leq r$, $i \neq j$. By Lemma 5.13, if $\Sigma_{p_1^{e_1}, p_2^{e_2}\ldots p_r^{e_r}}$ is planar, then $I$ must be planar. Since the vertices of $\Sigma_{p_j^2}$ are $\{p_j, 2p_j, \ldots, (p_j-1)p_j\}$, the mapping $qp_i \mapsto qp_j$ defines an isomorphism from $I$ to $\Sigma_{p_j^2}$. Since $I$ is planar, then $\Sigma_{p_j^2}$ must be planar. Therefore, by Proposition 5.7, $p_j \leq 7$ for all $1 \leq j \leq r$. ▲

Corollary 5.16. $\Sigma_n$ is only planar for

$$n = 4, 6, 8, 9, 10, 12, 14, 15, 21, 25, 35, 49.$$ 

For any planar graph, each component is isomorphic to one of the following:

Figure 4: $\Sigma_4$
Figure 5: $\Sigma_8$

Figure 6: $\Sigma_{12}$

Figure 7: $\Sigma_{25}$

Figure 8: $\Sigma_{49}$
Definition 5.17. A planar graph is outerplanar if every vertex of the graph lies on the boundary of the exterior region.

Definition 5.18. The graph \( K_{2,3} \) has five vertices. Two of the vertices \( a \) and \( b \) are adjacent to the three other vertices in the graph, \( c, d, \) and \( e. \) Also, \( a \sim b, c \sim d, \) \( c \sim e, \) and \( d \sim e. \)

Theorem 5.19. Let \( G \) be a graph. If \( G \) has a subgraph homeomorphic to \( K_{2,3}, \) then \( G \) is not outerplanar.

Proof. For a contradiction, assume \( G \) has a subgraph homeomorphic to \( K_{2,3} \) and \( G \) is outerplanar. Then it is possible to introduce a new vertex \( v \) inside the exterior region such that \( v \) is adjacent to \( c, d, \) and \( e \) and the resulting graph is planar. This would imply that \( K_{3,3} \) is planar, which is not true by Kuratowski’s Theorem. ▲

Proposition 5.20. The graph \( \Sigma_n \) is outerplanar if and only if
\[
n = 4, 6, 8, 9, 15, 25.
\]

Proof. If \( \Sigma_n \) is outerplanar, it must be planar. So we have a limited number of values for \( n \) to investigate. The graphs of \( \Sigma_4, \Sigma_8, \) and \( \Sigma_{25} \) are clearly outerplanar. The graph of \( \Sigma_{12} \) is not outerplanar because the vertices 10 and 2 are each adjacent to 8, 4, and 6. Thus it has a subgraph isomorphic to \( K_{2,3}. \) The graph of \( \Sigma_{19} \) is not outerplanar because the vertices 7 and 42 are each adjacent to 14, 21, and 28. Thus it has a subgraph isomorphic \( K_{2,3}. \) ▲

6 Automorphisms

Proposition 6.1. Let \( s \in \mathbb{Z}_n. \) Then the map \( M_s : V(\Sigma_n) \to V(\Sigma_n) \) defined by \( M_s(x) = sx \) induces an automorphism of \( \Sigma_n \) if and only if \( s \in \mathbb{Z}_n^*. \)

Proof. \( (\Leftarrow) \) Suppose \( u \in \mathbb{Z}_n \) is a unit and \( x, y \) are adjacent vertices. Then \( x + y \) is a vertex, so \( x + y \neq 0. \) Since \( u \) is a unit,
\[
\begin{align*}
ux + uy &= u(x + y) \\
&
\end{align*}
\]

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Thus, $ux \sim uy$. This shows that the map $M_u$ preserves edges.

We must show $M_u$ is one-to-one. Suppose that there exists $x, y \in V(\Sigma_n)$ such that $ux = uy$. Since $u \in \mathbb{Z}_n^*$, we have $u^{-1}(ux) = u^{-1}(uy)$. Therefore, $M_u$ is one-to-one, since $x = y$.

We do not need to show $M_u$ is onto because any one-to-one map from a finite set to itself is automatically onto.

Therefore $M_u$ a one-to-one correspondence between the vertices which preserves edges.

$(\Rightarrow)$ To prove the contrapositive, assume $z$ is not a unit. Thus $z$ is a zero divisor. Now suppose $z \in \mathbb{Z}_n$ is a zero divisor. Then there exists $t \neq 0$, such that $zt = 0$. Since $t \in V(\Sigma_n)$, $zt = 0$, which is not a vertex. Therefore $M_z$ is not an automorphism.

The following table shows all 50 automorphisms of $\Sigma_{16}$. Keep in mind that 8 is a dominating vertex and is adjacent to every other vertex in the graph. Since no other vertex is adjacent to everything else, then 8 will always map to itself. For that reason we will not include in the table $8 \mapsto 8.$

\begin{table}
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<th>4 → 10</th>
<th>12 → 6</th>
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8 Appendix: Glossary of Terms

In the following definitions, let $G = \Sigma_n$ be the graph with vertex set $V(\Sigma_n)$ and edge set $E(\Sigma_n)$.

**Adjacent:** Two vertices $u, v \in V(G)$ are said to be *adjacent* if $uv \in E(G)$.

**Associate:** Let $R$ be a ring, and $a, b \in R$. We say that $a$ is associate to $b$ if $a = ub$ for some unit $u$.

**Connected:** A graph $G$ is connected if there exists a $uv$-path between all pairs of distinct vertices $u$ and $v$ of $G$.

**Degree:** The degree of a vertex $v$ is the number of edges incident at it.

**Diameter:** The diameter of a graph is the greatest distance between any two vertices of a connected graph $G$ and is denoted $\text{diam}(G)$.

**Distance:** The distance between $u, v \in V(G)$ is the shortest length of a $uv$-path in $G$.

**Dominating vertex:** Any vertex, $v \in V(\Sigma_n)$, that is adjacent to every other vertex in $\Sigma_n$.

**Eulerian Graph:** A graph which has a closed trail containing every edge.

**Face:** A face is a portion of the graph which is bounded by edges and such that there is no edge running through the interior.

**Girth:** If a graph $G$ has a cycle, then the girth of $G$ is the length of the shortest cycle in $G$. The girth of a graph $G$ is denoted $gr(G)$.

**Graph:** A graph $G$ consists of a vertex set, $V(G)$, an edge set $E(G)$, and an association to each edge, $e \in E(G)$ of two vertices, called the endpoints of $e$.

**Incident:** An edge is incident at a vertex if that vertex is one of its endpoints.

**Minimum degree:** The minimum degree of a graph $G$ is the smallest degree of all
the vertices in a graph.

**Outerplanar**: A planar graph is outerplanar if every vertex of the graph lies on the boundary of the exterior region.

**Path**: A $uv$-path is a walk in a graph starting at a vertex $u$ and ending at a vertex $v$ in which no vertex is repeated

**Planar Graph**: A graph is planar if it can be drawn in the plane with no edge crossings.

**Subgraph**: Let $\Sigma_n$ be a graph. A subgraph of $\Sigma_n$ is a graph $H$ such that $V(H) \subseteq V(\Sigma_n)$ and $E(H) \subseteq E(\Sigma_n)$.

**Unit**: A unit is an element $u \in R$ that has a multiplicative inverse in $R$, i.e. there exists a $v \in R$ such that $uv = 1$ and $vu = 1$.

**Walk**: A walk in a graph $G$ is a sequence of vertices such that consecutive vertices in the sequence are adjacent.

**Zero divisor**: In a ring $R$, a zero divisor is an element $z \in R$ for which there exists $x \in R$, $x \neq 0$ such that $zx = 0$. 
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